Moreno-Socías Conjecture in Three Variables

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Introduction

When working with algebra, roots of the polynomials are frequently sought for. An ideal is a mathematical tool that helps us compute its roots by replacing complicated polynomials with simpler polynomials. In one of his papers, Guillermo Moreno-Socías conjectured that for every ideal, the initial ideal is a weakly reversed lexicographic ideal when ordered using graded reverse lexicographic ordering. The proof of the conjecture was given by Moreno-Socías in this paper. The conjecture has been proven in two variables using a different method, which is of interest, by Aguirre, Jarrah, Laubenbacher, et al.

Objectives

There was different stages of goals that we worked on during the summer. The Moreno-Socías conjecture is proven in two variables using this different method. This summer, we start attempting to prove for three variables. We started by finding solutions to the conjecture in specific cases, and rule out certain avenues of proof using this method, in the hope that future students could complete the proof.

Concepts

Let R = K[x_1, x_2, …, x_n] = K[x] be the polynomial ring in n variable over the infinite field K. A subset is an **ideal** if it satisfies:

1. If 0 is an element of the ideal.
2. If *f* and *g* are in the ideal, then *f* + *g* is in the ideal.
3. If *f* is in the ideal and *h* is an element of K[x_1, x_2, …, x_n], then *h f* is in the ideal.

Let α, β be elements of $\mathbb{Z}^n_{\geq 0}$. We say that $\alpha >_{grevlex} \beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i > |\beta| = \sum_{i=1}^{n} \beta_i$$

Or $|\alpha| = |\beta|$ and rightmost nonzero entry of $\alpha - \beta$ is negative.

A monomial ideal J is **weakly reverse lexicographic** if whenever x^a is a minimal generator of J, then every monomial of the same degree which precedes x^a in the reverse lexicographic term order must also be in J.

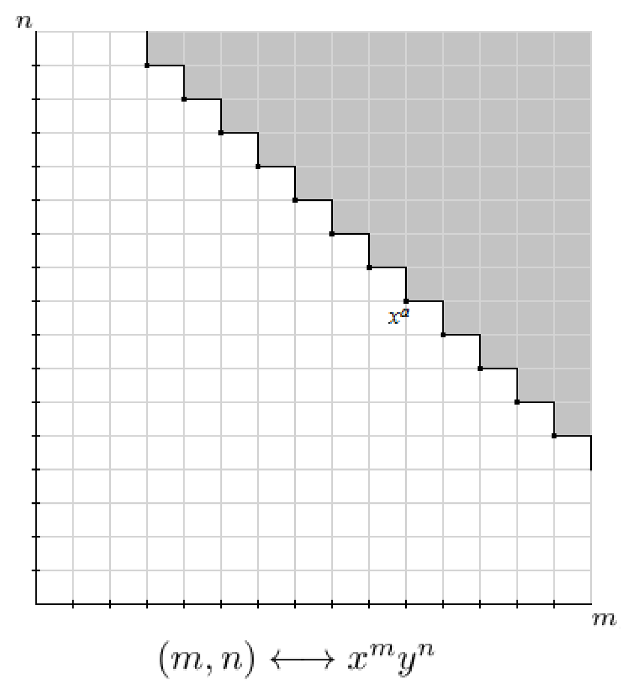

Figure 1. Weakly Reverse Lexicographic in 2-variables

Moreno-Socías Conjecture

Let d_1, …, d_n be an element of ℕ and I be a generic ideal generated by a sequence of polynomials f_1, …, f_n of degrees d_1, …, d_n and J = *in*(I), the initial ideal of I with respect to the graded reverse lexicographic order. Then J is weakly reverse lexicographic.

Methods

Initially, we studied Ideals, Varieties, and Algorithms by Cox, Little, and O'Shea to better understand the conjecture. The material was complex, but after a time investment, the material was absorbed. After this, we studied the proof of the two variables case of the conjecture, which detailed the method to be used. We then defined the polynomials in question for three variables case and generated a Gröbner basis from each set polynomials. We used computer calculations in Maple and, as calculations became protracted and eventually untenable, used Sage and Singular together to generate the Gröbner bases in a more direct way. We then endeavored to find the patterns in the leading monomials of the minimal generating polynomials Gröbner basis.

In doing so, we have found the patterns for f_1, f_2, and f_3 in its explicit form.

$$\begin{aligned}
f_1 = {} & a_{1,1,1}x^n + a_{1,1,2}x^{n-1}y + \ldots + a_{1,1,n}xy^{n-1} + a_{1,1,n+1}y^n \\
& + (a_{1,2,1}x^{n-1} + a_{1,2,2}x^{n-2}y + \ldots + a_{1,2,n}xy^{n-2} + a_{1,2,n+1}y^{n-1})z \\
& + (a_{1,3,1}x^{n-2} + a_{1,3,2}x^{n-3}y + \ldots + a_{1,3,n}xy^{n-3} + a_{1,3,n+1}y^{n-2})z^2 + \ldots \\
& + (a_{1,n,1}x)z^{n-1} + a_{1,n,1}z^n
\end{aligned}$$

$$\begin{aligned}
f_2 = \overline{f_2}^{f_1} = {} & a_{2,1,1}x^{n-1}y^{m-(n-1)} + a_{2,1,2}x^{n-2}y^{m-(n-2)} + \ldots + a_{2,1,n-1}xy^{m-1} + a_{2,1,n}y^m \\
& + (a_{2,2,1}x^{n-1}y^{(m-1)-(n-1)} + a_{2,2,2}x^{n-2}y^{(m-1)-(n-2)} \\
& + \ldots + a_{2,2,n-1}xy^{m-2} + a_{2,2,n}y^{m-1})z \\
& + (a_{2,3,1}x^{n-1}y^{(m-2)-(n-1)} + a_{2,3,2}x^{n-2}y^{(m-2)-(n-2)} \\
& + \ldots + a_{2,3,n-1}xy^{m-3} + a_{2,3,n}y^{m-2})z^2 \\
& + \ldots + (a_{2,m-n+2,1}x^{n-1} + a_{2,m-n+2,2}x^{n-2}y \\
& + \ldots + a_{2,m-n+2,n-1}xy^{n-2} + a_{2,m-n+2,n}y^{n-1})z^{m-n+1} \\
& + (a_{2,m-n+3,1}x^{n-2} + a_{2,m-n+3,2}x^{n-3}y \\
& + \ldots + a_{2,m-n+3,n-2}xy^{n-3} + a_{2,m-n+3,n-2}y^{n-2})z^{m-n+2} \\
& + (a_{2,m-n+4,1}x^{n-3} + a_{2,m-n+4,2}x^{n-4}y \\
& + \ldots + a_{2,m-n+4,n-3}xy^{n-4} + a_{2,m-n+4,n-3}y^{n-3})z^{m-n+3} \\
& + \ldots + (a_{2,m,1}x + a_{2,m,2}y)z^{m-1} + a_{2,m+1,1}z^m
\end{aligned}$$

$$\begin{aligned}
f_3 = \overline{f_3}^{F} = {} & a_{3,1,1}x^{n-2}y^{l-(n-2)} + a_{3,1,2}x^{n-3}y^{l-(n-3)} + \ldots + a_{3,1,n-2}xy^{l-1} + a_{3,1,n-1}y^l \\
& + (a_{3,2,1}x^{n-2}y^{(l-1)-(n-2)} + a_{3,2,2}x^{n-3}y^{(l-1)-(n-3)} + \ldots + a_{3,2,n-2}xy^{l-2} + a_{3,2,n-1}y^{l-1})z \\
& + (a_{3,3,1}x^{n-2}y^{(l-2)-(n-2)} + a_{3,3,2}x^{n-3}y^{(l-2)-(n-3)} + \ldots + a_{3,3,n-2}xy^{l-3} + a_{3,3,n-1}y^{l-2})z^2 \\
& + \ldots + (a_{3,l-m+1,1}x^{n-2}y^{m-(n-2)} + a_{3,l-m+1,2}x^{n-3}y^{m-(n-3)} \\
& + \ldots + a_{3,l-m+1,n-2}xy^{m-1} + a_{3,l-m+1,n-1}y^m)z^{l-m} \\
& + (a_{3,l-m+2,1}x^{n-1}y^{(m-1)-(n-1)} + a_{3,l-m+2,2}x^{n-2}y^{(m-1)-(n-2)} \\
& + \ldots + a_{3,l-m+2,n-1}xy^{m-2} + a_{3,l-m+2,n}y^{m-1})z^{l-m+1} \\
& + \ldots + (a_{3,l-n+2,1}x^{n-1} + a_{3,l-n+2,2}x^{n-2}y \\
& + \ldots + a_{3,l-n+2,n-1}xy^{n-2} + a_{3,l-n+2,n}y^{n-1})z^{l-n+1} \\
& + (a_{3,l-n+3,1}x^{n-2} + a_{3,l-n+3,2}x^{n-3}y \\
& + \ldots + a_{3,l-n+3,n-2}xy^{n-3} + a_{3,l-n+3,n-1}y^{n-2})z^{l-n+2} \\
& + \ldots + (a_{3,l,1}x + a_{3,l,2}y)z^{l-1} + a_{3,l+1,1}z^l
\end{aligned}$$

Results and Conclusions

We made a useful program using Singular to aid us in generating the Gröbner basis, that enabled us to discover the full Gröbner basis of several specific cases, which could perhaps go further with some more work. Alongside that, we have found the general case for the first three polynomials of the Gröbner basis. Finally, we found the minimal Gröbner basis for several specific cases, but have found no general case as of yet.
We achieved several important steps down the road towards it, which was our primary goal. The groundwork we have laid should allow a more fully fruitful future research experience, the experience gained has been very valuable, and we have gained a new appreciation for this type of research.

Sources

Cox, David, et al. *Ideals, Varieities, and Algorithms.* New York, NY. 2008.